UDC 539.3

BENDING OF A SYSTEM OF STRIP PLATES LYING ON AN ELASTIC HALF-SPACE

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V. S. PROTSENKO
(Khar'kov)
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A method of solving a problem of contact between a finite system of infinite strip plates of varying width and an elastic half-space, is given. Friction within the region of contact is neglected. The problem is reduced to a set of infinite systems of algebraic equations which can be solved using the method of reduction.

1. A system consisting of m strip plates lies on an uniform elastic half-space parallel to the y-axis. The problem reduces to the solution of the following system of equations:

$$\frac{1-\nu^2}{\pi E}\sum_{i=1}^{m}\int_{L_i}\bar{r}_i(\xi,\lambda)K_0(\lambda|x-\xi|)\,d\xi=\bar{w}_i(x,\lambda) \tag{1.1}$$

$$D_i\left(\frac{d^2}{dx^2}-\lambda^2\right)^2\bar{w}_i(x,\lambda)-\bar{g}_i(x,\lambda)+\bar{r}_i(x,\lambda)=0$$

$$(x\in L_i=[c_i,d_i],\ i=1,2,\ldots,m)$$

Here $\partial\Omega_i$ denotes the region of contact of the i-th plate with the half-space, D_i denotes the rigidity of the plate, and w_i, g_i and r_i are, respectively, the deflection, the load and the reaction of the support. We assume that the plate edges are free and, that the functions g_i, w_i and r_i can be represented by a Fourier integral (where λ is the transformation parameter)

$$\mathbf{a}_i(x,\lambda)=F^{-1}[\mathbf{b}_i(x,\lambda)],\quad \mathbf{b}_i(x,\lambda)=F[\mathbf{a}_i(x,y)] \tag{1.2}$$

$$\mathbf{a}_i=\{g_i,w_i,r_i\},\quad \mathbf{b}_i=\{\bar{g}_i,\bar{w}_i,\bar{r}_i\}$$

Taking (1.2) into account we obtain, from (1.1),

$$\frac{1-\nu^2}{\pi E}\sum_{i=1}^{m}\iint_{\partial\Omega_i}\frac{r(\zeta,\eta)\,d\zeta\,d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}=w_i(x,y) \tag{1.3}$$

$$D_i\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w_i(x,y)=g_i(x,y)-r_i(x,y)\ ((x,y)\in\partial\Omega_i,\ i=1,2,\ldots,m) \tag{1.4}$$

As we know [1], the solution of Eq. (1.3) can be reduced to that of the following boundary value problem:

$$\Delta\varphi-\lambda^2\varphi=0 \tag{1.5}$$

$$\varphi(x,\lambda,0)=\bar{w}_i(x,\lambda),\quad x\in L_i;\quad \frac{\partial\varphi}{\partial z}\Big|_{z=0}=0\ \ x\,\overline{\in}\,L_i$$

for the potential $\varphi(x, \lambda, z) = \sum_{i=1}^{m} \int_{L_i} \bar{r}_i(\xi, \lambda) K_0(\lambda \sqrt{(x-\xi)^2 + z^2})\, d\xi$

vanishing at infinity.

Having solved the boundary value problem (1.5), we can find the function $\bar{r}_i(x,\lambda)$ from the formula

$$\bar{r}_i(x, \lambda) = \frac{E}{2(1-\nu^2)} \frac{\partial \varphi}{\partial z}\Big|_{z=0}, \quad x \in L_i \; (i = 1, 2, \ldots, m)$$

Let us introduce the local Cartesian coordinates $\bar{x}_i$, o_i, z_i, placing their origins in the middle of the segments L_i $(i = 1, 2, \ldots, m)$, and let us perform the variable substitution $\bar{x}_i = a_i x_i$ and $\bar{z}_i = a_i z_i$ where $2a_i$ is the width of the segment L_i. Passing now in each local coordinate system x_i, o_i, z_i to the elliptic cylinder coordinates according to the formulas

$$x_i = \operatorname{ch} \xi_i \cos \eta_i, \quad z_i = \operatorname{sh} \xi_i \sin \eta_i$$
$$(0 \leqslant \eta_i \leqslant 2\pi, \; 0 \leqslant \xi_i < \infty)$$

we find, in accordance with [1],

$$\bar{r}_i(x_i, \lambda) = \frac{E (a_i^2 - x_i^2)^{-\frac{1}{2}}}{2(1-\nu^2)} \sum_{n=0}^{\infty} r_n^{(i)} \operatorname{Fek}_n'(0, -q_i) \operatorname{ce}_n(\eta_i, -q_i) \tag{1.6}$$

The coefficients $r_n^{(i)}$ are found from the set of infinite systems

$$r_k^{(i)} + \sum_{s \neq i}^{m} \sum_{n=0}^{\infty} r_n^{(s)} T_n^{(k)}(s, i) = \frac{\gamma_k^{(i)}}{\operatorname{Fek}_k(0, -q_i)}$$
$$T_n^{(k)}(s, i) = \frac{Q_k^{(n)}(s, i) \operatorname{Ce}_k(0, -q_s)}{\operatorname{Fek}_k(0, -q_i)} \tag{1.7}$$

where $\gamma_k^{(i)}$ are the coefficients of expansion of the function $\bar{w}_i$ into series in terms of the Mathieu functions on L_i

$$\bar{w}_i(x, \lambda) = \sum_{n=0}^{\infty} \gamma_n^{(i)} \operatorname{ce}_n(\eta_i, -q_i) \tag{1.8}$$

2. In dimensionless local coordinates x_i, z_i, Eq. (1.4) becomes

$$D_i^* \left(\frac{d^2}{dx_i^2} - \lambda_i^2\right)^2 \bar{w}_i(x_i, \lambda) - \bar{g}_i(x_i, \lambda) + \bar{r}_i(x_i, \lambda) = 0 \tag{2.1}$$
$$(|x_i| \leqslant 1, \quad D_i^* = D_i / a_i^4, \quad \lambda_i = \lambda a_i)$$

Let $G(x, \xi)$ be the Green function of the boundary value problem

$$(d^2/dx^2 - \lambda^2)^2 y = 0, \quad y(\pm 1) = y''(\pm 1) = 0 \tag{2.2}$$

Then the solution of (2.1) with conditions (2.2) can be written in the form of an integral

$$\bar{w}_i(x_i, \lambda) = D_i^{*-1} \int_{-1}^{1} G_i(x_i, \xi) [\bar{g}_i(\xi, \lambda) - \bar{r}_i(\xi, \lambda)]\, d\xi \tag{2.3}$$

provided that the function $\bar{r}_i(x_i, \lambda)$ is, for the time being, assumed known.

The function $\bar{w}_i(x_i, \lambda)$ must satisfy the boundary conditions (the edges are free)

$$\frac{d^2\bar{w}_i}{dx_i^2} - \nu_0^{(i)}\lambda_i^2\bar{w}_i = \frac{d^3\bar{w}_i}{dx_i^3} - \lambda_i^2(2-\nu_0^{(i)})\frac{d\bar{w}_i}{dx_i} = 0, \; x_i = \pm 1 \tag{2.4}$$

where $\nu_0^{(i)}$ is the Poisson's ratio for the plate.

The above conditions can be easily realized by adding the general solution of the homogeneous equation (2.2) to the solution given by (2.3) and satisfying the conditions (2.2), and requiring that the four conditions of (2.4) are satisfied. In connection with this, the boundary conditions for the function $\bar{w}_i(x_i, \lambda)$ will be taken, from now on, in the form (2.2).

Using (1.6) and (1.8) we obtain (2.3)

$$\sum_{n=0}^{\infty} \gamma_n^{(i)} \operatorname{ce}_n(\eta_i, -q_i) = -k_0^{(i)} \int_{-1}^{1} G_i(\xi, x_i) \times \tag{2.5}$$

$$\sum_{n=0}^{\infty} r_n^{(i)} \operatorname{Fek}_n'(0, -q_i) \frac{\operatorname{ce}_n(t, -q_i)}{|\sin t|} d\xi + f_i(\eta_i)$$

$(\xi = \cos t, \; x_i = \cos \eta_i)$

$$k_0^{(i)} = E[D_i^*(1-\nu^2)2a_i]^{-1}, \quad f_i(\eta_i) = D_i^{*-1}\int_{-1}^{1} G_i(\xi, x_i)\bar{g}_i(\xi, \lambda)\, d\xi \tag{2.6}$$

Multiplying Eq. (2.5) by $\operatorname{ce}_k(\eta_i, -q_i)$ and integrating over the interval $(0, \pi)$, we obtain

$$\pi_k\gamma_k^{(i)} + \sum_{n=0}^{\infty} R_n^{(i)} K_n^{(k)}(i) = \alpha_k^{(i)} \tag{2.7}$$

$$(\pi_0 = \pi, \; \pi_k = {}^1/_2\pi, \; k \geqslant 1, \; R_n^{(i)} = r_n^{(i)} \operatorname{Fek}_n(0, -q_i))$$

The matrix coefficients and free terms in (2.7) are determined by the formulas

$$K_n^{(k)}(i) = k_0^{(i)} \frac{\operatorname{Fek}_n'(0, -q_i)}{\operatorname{Fek}_n(0, -q_i)} \int_0^{\pi} \operatorname{ce}_k(\eta, -q_i)\, d\eta \times \tag{2.8}$$

$$\int_{-1}^{1} G_i(\xi, x) \frac{\operatorname{ce}_n(t, -q_i)}{|\sin t|} d\xi$$

$$\alpha_k^{(i)} = D_i^{*-1}\int_0^{\pi} \operatorname{ce}_k(\eta, -q_i)\, d\eta \int_{-1}^{1} G_i(\xi, x)\bar{g}_i(\xi, \lambda)\, d\xi \tag{2.9}$$

and we transform the set of systems (1.7),(2.7) to the form

$$R_k^{(i)} + \sum_{s\neq i}^{m} \sum_{n=0}^{\infty} R_n^{(s)} T_n^{*(k)}(s, i) + \sum_{n=0}^{\infty} R_n^{(i)} K_n^{(k)}(i) = \alpha_k^{(i)}, \tag{2.10}$$

$$T_n^{*(k)}(s, i) = \frac{Q_k^{(n)}(s, i) \operatorname{Ce}_k(0, -q_s)}{\operatorname{Fek}_n(0, -q_s)}$$

Thus we have reduced our problem to that of solving a set of m infinite systems of linear algebraic equations for the coefficients $R_n^{(i)}$.

3. Investigation of the properties of the system (2.10) requires additional formulas from the theory of Mathieu functions. From the results of [2] it follows that

$$Q_k^{(n)}(s,i) \sim C^{(n)}(i)[C^{(k)}(s)]^{-1} K_{k+n}(\lambda R_{si}) \quad (n, k \to \infty)$$

$$C^{(2n)}(i) = A_0^{(2n)}(q_i), \quad C^{(2n+1)}(i) = B_1^{(2n+1)}(q_i)\frac{1}{2n+1}$$

Here $K_n(t)$ is the Macdonald function and R_{si} is the distance separating the centers of the Cartesian coordinate systems with indices s and i. Taking into account the asymptotic formulas [3] for the coefficients $A_0^{(2n)}$), $B_1^{(2n+1)}$ and function $K_n(t)$, we obtain

$$T_n^{*(k)}(s,i) \sim \text{const}_1 \left(\frac{a_i}{R_{si}}\right)^k \left(\frac{a_s}{R_{si}}\right)^n \frac{(n+k)!}{n!k!} \tag{3.1}$$

The properties of the matrix $(K_n^{(k)})$ and of the free terms depend on the properties of the Green function $G_i(\xi, x)$. The latter can be written in the form of an absolutely and uniformly convergent bilinear expansion [4] in terms of the eigen-functions of the boundary value problem

$$(d^2/dx^2 - \lambda^2)^2 y - \mu y = 0, \quad y(\pm 1) = y''(\pm 1) = 0$$

The expansion has the form

$$G(\xi, x) = \sum_{k=1}^{\infty} \frac{\sin p_{2k}x \sin p_{2k}\xi}{\mu_{2k}} + \sum_{k=0}^{\infty} \frac{\cos p_{2k+1}x \cos p_{2k+1}\xi}{\mu_{2k+1}} \tag{3.2}$$

$$\mu_k = (p_k^2 + \lambda^2)^2, \quad p_k = {}^1/_2 \pi k$$

The function (3.2) is given in [4] in the form of a dual properly converging series in terms of the Tchebycheff polynomials

$$G(\xi, x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} T_i(x) T_j(\xi) \tag{3.3}$$

and its coefficients are determined by the formulas

$$a_{2m,\,2n} = \lambda_{2m,\,2n} \sum_{k=0}^{\infty} \frac{1}{\mu_{2k+1}} J_{2m}(p_{2k+1}) J_{2n}(p_{2k+1})$$

$$a_{2m+1,\,2n+1} = \frac{1}{4}(-1)^{m+n} \sum_{k=1}^{\infty} \frac{1}{\mu_{2k}} J_{2m+1}(p_{2k}) J_{2n+1}(p_{2k})$$

$$\lambda_{0,0} = {}^1/_4, \quad \lambda_{2n,0} = {}^1/_2(-1)^n, \quad \lambda_{2m,2n} = {}^1/_4(-1)^{m+n}$$

The author also proves that the following inequality holds:

$$|a_{m,\,n}| \leqslant \frac{\text{const}_2}{mn \max(n, m)} \quad (m \geqslant 2,\ n \geqslant 2)$$

and this yields the inequality

$$|a_{m,\,n}| \leqslant \frac{\text{const}_3}{mn(m+n)} \quad (m \geqslant 2,\ n \geqslant 2) \tag{3.4}$$

Substituting the series (3.3) into (2.8) and computing the corresponding integrals, we obtain

$$K_n^{(k)}(i) = k_0^{(i)} \frac{\operatorname{Fek}_n'(0, -q_i)}{\operatorname{Fek}_n(0, -q_i)} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \pi_m \pi_j C_m^{(k)}(i)\, C_j^{(n)}(i)\, a_{mj} \tag{3.5}$$

$$C_{2r}^{(2n)}(i) = (-1)^{r+n} A_{2r}^{(2n)}(q_i), \quad C_{2r+1}^{(2n+1)}(i) = (-1)^{r+n} B_{2r+1}^{(2n+1)}(q_i)$$

From (3.5) it follows that for large values of the indices n and k

$$K_n^{(k)}(i) \sim \frac{\text{const}_4}{nk(n+k)} \frac{\operatorname{Fek}_n(0, -q_i)}{\operatorname{Fek}_n(0, -q_i)} \sim \frac{\text{const}_5}{k(n+k)} \tag{3.6}$$

The asymptotic formulas (3.1) and (3.6) guarantee the convergence of the series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |K_n^{(k)}(i)|^2, \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |T_n^{*(k)}(s, i)|^2$$

Consequently, the system (2.10) generates in l^2 a completely continuous operator [5]. Taking into account the expansion (3.3), we can write the formulas for the free terms (2.9) as

$$\alpha_{2k}^{(i)} = \pi D_i^{*-1}(-1)^k \sum_{m=0}^{\infty} \frac{a_m^{(i)}}{\mu_{2m+1}} \sum_{r=0}^{\infty} A_{2r}^{(2k)} J_{2r}\left(\pi m + \frac{\pi}{2}\right) \tag{3.7}$$

$$\alpha_{2k+1}^{(i)} = \pi D_i^{*-1}(-1)^k \sum_{m=1}^{\infty} \frac{b_m^{(i)}}{\mu_{2m}} \sum_{r=0}^{\infty} B_{2r+1}^{(2k+1)} J_{2r+1}(m\pi)$$

$$b_m^{(i)} = \int_{-1}^{1} \bar{g}_i(\xi, \lambda) \sin(m\pi\xi)\, d\xi$$

$$a_m^{(i)} = \int_{-1}^{1} \bar{g}_i(\xi, \lambda) \cos\frac{\pi}{2}(2m+1)\xi d\xi \quad (m \geqslant 0)$$

From (3.7) it follows that even when the load on the plate is concentrated, the coefficients $\alpha_k^{(i)} = O(k^{-2})$ as $k \to \infty$.

The infinite system (2.10) with the operator completely continuous in l^2 and free terms $\alpha_k^{(i)} \in l^2$, can be solved using the method of reduction. The solution of such a system which also belongs to l^2 can be obtained with any required accuracy. More rigorous analysis of systems (2.7), (2.10) enables us to conclude that the coefficients $R_k^{(i)}$ and $\gamma_k^{(i)}$ are of the order not higher than k^{-2} when $k \to \infty$. Thus the series for $\overline{w}_i$ always converges uniformly and absolutely and represents a continuous function.

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